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Connections between the Support and Linear Independence of Refinable Distributions

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The purpose of this paper is to study the relationships between the support of a refinable distribution ϕ and the global and local linear independence of the integer translates of ϕ. It has been shown elsewhere that a compactly supported distribution ϕ has globally independent integer translates if and only if ϕ has minimal convex support. However, such a distribution may have "holes" in its support. By insisting that $\phi \in L^2(\mathbb{R})$ and generates a multiresolution analysis, Lemarié and Malgouyres have ensured that no such holes can occur. In this article we generalize this result to refinable distributions. We also give a result on the local linear independence of the integer translates of ϕ. We work with integer dilation factor $N \geqslant 2$ throughout this paper.

1. INTRODUCTION AND BASIC CONCEPTS

This paper investigates the support of a refinable distribution ϕ and the global and local linear independence of the integer translates of ϕ. These ideas are fundamental in wavelet theory and have been studied considerably elsewhere (e.g., [3, 6, 11, 7, 9]). We say that a distribution ϕ is *refinable* with dilation factor $N \geqslant 2$ if there exist scalars p_k for which ϕ satisfies

$$\phi(x) = \sum_{k \in \mathbb{Z}} p_k \phi(Nx - k). \tag{1}$$

We use the term *refinable* rather than *scaling* distribution to emphasize that we do not assume that ϕ is an L^p function; ϕ need not generate a multiresolution analysis (MRA) for our results (see [4] for a detailed discussion of MRAs). We will assume throughout that ϕ is compactly supported; see [5] for a discussion of this condition.

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Deslauriers and Dubuc [6] showed that if ϕ is a refinable, compactly supported distribution and $p_k = 0$ for $k < 0$ and for $k > M$ in (1) then ϕ has its support in $[0, M/(N-1)]$. Throughout this paper, we say that the *convex support* of ϕ is the closure of the convex hull of $\text{supp}(\phi)$. For $N = 2$, Chui and Wang [3] have shown that the convex support of ϕ is $[0, M/(N-1)]$, and their proof is easily generalized to the case $N > 2$.

It has been shown that the convex support of a compactly supported distribution ϕ is related to the linear independence of the integer translates of ϕ. Before going on, let us make some of our terminology more precise. Throughout this paper, we denote the set of integer translates of ϕ by

$$T_\phi = \{\phi(\cdot - n)\}_{n \in \mathbb{Z}}. \tag{2}$$

We say that T_ϕ is *globally linearly independent* (GLI) if $\sum_{n \in \mathbb{Z}} c_n \phi(\cdot - n) = 0$ and $c_n \in \mathbb{C}$ implies that $c_n = 0 \; \forall n$. This condition is also referred to as "algebraically linearly independent" by some authors.

We also need to define the concept of "support" more carefully, since it has been used in a variety of ways in the literature. For $\varphi \in \mathcal{D}(\mathbb{R})$, the space of test functions (compactly supported C^∞-functions), we define the support of φ in the usual way, as the closure of $\{x \in \mathbb{R} : \varphi(x) \neq 0\}$. For distributions, we use the following standard definition of support (see, e.g., [13]). Recall that $\mathcal{D}'(\mathbb{R})$, the dual of $\mathcal{D}(\mathbb{R})$, is the space of *Schwartz distributions*.

Definition 1. Let $f \in \mathcal{D}'(\mathbb{R})$, W an open subset of $\mathbb{R}$.

1. We say f *vanishes on* W if $\langle f, \varphi \rangle = 0$ for all test functions $\varphi \in \mathcal{D}(W)$.

2. Let U be the union of all open sets $W \subset \mathbb{R}$ on which f vanishes. We define the *support* of f by

$$\text{supp } f = U^c,$$

the complement of U in $\mathbb{R}$.

3. We say f is of *minimal support* in a space S if $f \neq 0$ and $m(\text{supp } f) \leqslant m(\text{supp } g)$ for all nonzero $g \in S$, where m denotes Lebesgue measure.

Note that if f is a continuous function, then the support of f is the closure of $\{x \in \mathbb{R} : f(x) \neq 0\}$, which coincides with Definition 1. Also note that it is possible for the support of f to have "holes" in it.

The following can easily be deduced from Result 3.2 of Ron in [12].

THEOREM 2. *Let ϕ be a compactly supported distribution. A distribution f in the principal shift-invariant space $S = \text{span}\, T_\phi$ has minimal convex support if and only if T_f is GLI.*

When the generator of space S is not minimally supported, Jia has provided in Theorem 5.1 of [8] a method to find the minimally convex supported refinable function that generates S (the proof is for dilation factor $N=2$, but the proof is straightforward for $N>2$). For greater clarity, we will tend to speak in terms of the GLI of T_ϕ rather than the minimal convex support of ϕ.

Two natural questions to ask at this point are whether the support of ϕ could actually have any "holes" in it, and what conditions can be placed on ϕ to ensure that no such holes exist. Certainly, it is not hard to conjure up an example of a compactly supported, refinable function which has as its support the union of disjoint closed intervals, e.g., $\phi(x) = \chi_{[0,1)} + \chi_{[2,3)}$. However, observe that T_ϕ is not GLI for this ϕ. Lemarié and Malgouyres provide a nice proof of the following result for L^2 functions and dilation factor $N=2$ in [11].

THEOREM 3. *Suppose $\phi \in L^2(\mathbb{R})$ has minimal (and compact) convex support in* span T_ϕ *and generates a (two-scale) multiresolution analysis. Then*

1. *the support of ϕ is an interval, and*
2. *the restrictions to* $[0,1]$ *of the integer translates of ϕ must be linearly independent.*

A few remarks are in order. Since the multiresolution analysis (MRA) hypothesis is crucial in their proof, it is not obvious that the result holds for a distribution ϕ, even for dilation factor $N=2$. If $N>2$, then there are more than two wavelet generators implied by the multiresolution analysis (see [4]). This would certainly complicate the approach of Lemarié and Malgouyres.

We also note that, in general, the second conclusion of Theorem 3 is weaker than obtaining local linear independence on arbitrary intervals. To be precise, we follow Goodman and Lee [7] and say that T_ϕ is *locally linearly independent* on a nontrivial interval (a,b) if $\sum_{k\in\mathbb{Z}} d_k \phi(\cdot - k) = 0$ on (a,b) and $d_k \in \mathbb{C}$ implies that $d_k = 0$ for all k for which $\phi(\cdot - k)$ is not identically zero on (a,b). When T_ϕ is locally linearly independent on arbitrary intervals, we simply say that T_ϕ is *locally linearly independent*. It is obvious that if T_ϕ is locally linearly independent, then it is also GLI, but the converse is not true. To further illustrate the connections between the concepts discussed here, consider the example

$$\varphi(t) = t\chi_{[0,1)}(t) + \chi_{[1,4)}(t).$$

Note that T_φ is GLI but not locally linearly independent. Also observe that φ does not have minimal support in span T_φ. The function $\psi(x) = \varphi(x) - \varphi(x-1)$ has minimal support in span T_φ, but its support has a "hole" in it, and T_ψ is not GLI. However, φ in this example is not refinable. We shall see that these types of linear independence of T_ϕ prove to be equivalent when ϕ is refinable and compactly supported.

There are two goals of this paper. The first is to show that the $\phi \in L^2(\mathbb{R})$ and MRA hypotheses in Theorem 3 can be replaced by requiring ϕ to be a refinable and compactly supported distribution. Second, we will prove this for any integer dilation factor $N \geqslant 2$.

We are now in a position to state our main theorem.

THEOREM 4. *Suppose that $\phi \in \mathscr{D}'(\mathbb{R})$ is a refinable, compactly supported distribution satisfying*

$$\phi(x) = \sum_{j=0}^{M} p_j \phi(Nx - j), \qquad p_0 p_M \neq 0, \tag{3}$$

and such that $T_\phi = \{\phi(\cdot - n)\}_{n \in \mathbb{Z}}$ is GLI. Then

1. $\operatorname{supp}(\phi) = [0, M/(N-1)]$, *and*
2. T_ϕ *is locally linearly independent.*

The rest of the paper is outlined as follows. In Section 2 we give some results crucial for proving Theorem 4 in the case $N > 2$. In Section 3 we state and prove a chain of lemmata that leads to a proof of Theorem 4.

2. AN INTERPOLATION RESULT

In this section, we shall prove an interpolation result for a system of generalized exponential functions. For this purpose we first introduce a result for generalized Vandermonde matrix.

For $z \in \mathbb{C}$, let $X(z) = (1, z, ..., z^{n-1})^T$ be an n-dimensional vector, and let $X^{(j)}(z)$ be its jth derivative. Assume that $\{\lambda_1, ..., \lambda_s\} \in \mathbb{C}$ and $\{n_1, ..., n_s\} \in \mathbb{N}$, with $\sum_{j=1}^{s} n_j = n$. For $1 \leqslant j \leqslant s$, define $n \times (n_j + 1)$ matrices U_j by

$$U_i = [X(\lambda_1), ..., X^{(n_j)}(\lambda_1)]$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \lambda_i & 1 & \cdots & 0 \\ \lambda_i^2 & 2\lambda_i & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \lambda_i^{n-1} & (n-1)\,\lambda_i^{n-2} & \cdots & (n-1)\cdots(n-n_j)\,\lambda_i^{n-1-n_j} \end{bmatrix}.$$

Then the $n \times n$ matrix

$$\mathbf{U} = [U_1, ..., U_s] \tag{4}$$

is called a generalized Vandermonde matrix.

The following result for a Vandermonde matrix is known (see [10]).

LEMMA 5. *The generalized Vandermonde matrix* $\mathbf{U}$ *in* (4) *is nonsingular if and only if* $\lambda_i \neq \lambda_j$ *for* $i \neq j$.

We also need the following.

LEMMA 6. *For* $1 \leqslant k$, *and* $j \in \mathbb{Z}$, *define the integers* p_k^j *by the induction that* $p_1^j = \delta_{1j}$, $j \in \mathbb{Z}$, *and* $p_k^j = p_{k-1}^{j-1} + j p_{k-1}^j$. *Let* $P(m, j) = m(m-1) \cdots (m+1-j)$, $1 \leqslant j \leqslant m$. *Then*

$$m^k = \sum_{j=1}^{k} P(m, j)\, p_k^j. \tag{5}$$

Proof. We prove the lemma by mathematical induction. When $k = 1$, (5) is trivial. Under the assumption that (5) is true for $k-1$, we prove it is true for k. In fact, by $p_k^1 = 1$, $p_k^k = 1$, and $p_k^l = 0$, $l \leqslant 0$ or $l > k$, we have

$$\begin{aligned}
\sum_{j=1}^{k} P(m, j)\, p_k^j &= \sum_{j=1}^{k} P(m, j)(p_{k-1}^{j-1} + j p_{k-1}^j) \\
&= \sum_{j=1}^{k} P(m, j-1)(m+1-j)\, p_{k-1}^{j-1} + \sum_{j=1}^{k} P(m, j)\, j p_{k-1}^j \\
&= \sum_{j=1}^{k-1} P(m, j)(m-j)\, p_{k-1}^j + \sum_{j=1}^{k-1} P(m, j)\, j p_{k-1}^j \\
&= m \sum_{j=1}^{k-1} P(m, j)\, p_{k-1}^j \\
&= m^k
\end{aligned}$$

The lemma is proved. ∎

LEMMA 7. *For* $1 \leqslant i \leqslant s$, *define* $n \times (n_i + 1)$ *matrices* V_i *by*

$$V_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \lambda_i & \lambda_i & \cdots & \lambda_i \\ \lambda_i^2 & 2\lambda_i^2 & \cdots & (2)^{n_i} \lambda_i^2 \\ \vdots & \vdots & & \vdots \\ \lambda_i^{n-1} & (n-1)\,\lambda_i^{n-1} & \cdots & (n-1)^{n_i}\,\lambda_i^{n-1} \end{bmatrix},$$

where $n=\sum_{i=1}^{s}(n_i+1)$. *The matrix*

$$\mathbf{V}=[V_1, ..., V_s] \tag{6}$$

is nonsingular if and only if

$$\lambda_i \neq 0 \ \forall i, \qquad \text{and} \qquad \lambda_i \neq \lambda_j \qquad \text{for} \quad i \neq j. \tag{7}$$

Proof. Denote the kth column in V_i by $\mathbf{v}^k$ and the jth column in U_i by $\mathbf{u}^j$. By Lemma 6, $\mathbf{v}_k=\sum_{j=1}^{k} p_k^j \lambda^j \mathbf{u}_j$, $1 \leqslant k \leqslant n_i$. Therefore, $\det \mathbf{V}=\prod_{i=1}^{s} \lambda_i^{(n_i(n_i+1))/2} \det \mathbf{U}$. By Lemma 4, we obtain the sufficient and necessary condition (7). ∎

Corollary 8. *Let* $\mathcal{S}=\text{span}\{(\lambda_1)^x, x(\lambda_1)^x, ..., x^{n_1-1}(\lambda_1)^x, ..., \lambda_s^x, x(\lambda_s)^x, ..., x^{n_s-1}(\lambda_s)^x\}$, *where* $\lambda_i \neq 0$ $\forall i$, $\lambda_i \neq \lambda_j$ *for* $i \neq j$, *and* $n=\sum_{i=1}^{s} n_i$. *If* $f \in \mathcal{S}$ *and* $f(k)=0$ *for* $k=0, ..., n-1$, *then* $f(x)=0$ *for all* $x \in \mathbb{R}$.

Proof. Let $f(x)=\sum_{i=1}^{s}\sum_{t=0}^{n_i-1} a_{it} x^t(\lambda_i)^x \in \mathcal{S}$. Define $\mathbf{a}$ by

$$\mathbf{a}=(a_{10}, ..., a_{1n_1-1}, ..., a_{s0}, ..., a_{sn_s-1})^T.$$

Then $f(k)=0$ for $k=0, ..., n-1$ can be rewritten as $\mathbf{Va}=\mathbf{0}$ where $\mathbf{V}$ is defined in (6). From Lemma 7 we know that $\mathbf{V}$ is nonsingular, whence $\mathbf{a}=\mathbf{0}$. ∎

The following "interpolation" proposition is a key ingredient in the proof of Lemma 15.

Proposition 9. *Let*

$$f(x)=\sum_{i=1}^{s}\sum_{t=0}^{n_i-1} a_{it} x^t(\lambda_i)^x$$

$$g(x)=\sum_{i=1}^{r}\sum_{t=0}^{m_i-1} b_{it} x^t(\mu_i)^x.$$

Set $p=\sum_{i=1}^{s} n_i$, $q=\sum_{i=1}^{r} m_i$. *Assume that* $p, q \in \mathbb{N}$, $p \geqslant q$,

$$\lambda_i, \mu_i \neq 0 \qquad \forall i, \qquad \text{and} \qquad \lambda_i \neq \lambda_j, \qquad \mu_i \neq \mu_j \qquad \text{for} \quad i \neq j. \tag{8}$$

Then

$$f(k)=g(k) \qquad \text{for} \quad k=0, ..., 2p-1 \tag{9}$$

if and only if

$$f(x)=g(x) \qquad \forall x. \tag{10}$$

Proof. (⇐) This direction is trivial.

(⇒) Define $C(x)$ and $\mathscr{S}$ by

$$C(x)=\sum_{i=1}^{s}\sum_{t=0}^{n_i-1} a_{it}\,x^t(\lambda_i)^x-\sum_{i=1}^{r}\sum_{t=0}^{m_i-1} b_{it}\,x^t(\mu_i)^x \tag{11}$$

and

$$\mathscr{S}=\operatorname{span}\{\lambda_1^x, ..., x^{n_1-1}\lambda_1^x, ..., \lambda_s^x, ..., x^{n_s-1}\lambda_s, \mu_1^x, ..., x^{m_1-1}\mu_1^x, ..., \mu_r^x, ..., x^{m_r-1}\mu_r^x\}. \tag{12}$$

It is clear that $\dim \mathscr{S}\leqslant p+q\leqslant 2p$. Now $C(x)\in\mathscr{S}$ and $C(k)=0$ for $k=0, ..., 2p-1$, so Corollary 8 yields $C(x)=0$ for all x. The conclusion (10) follows. ∎

3. SOME BASIC LEMMAS

To prove Theorem 4, we shall develop a sequence of lemmata in this section. A crucial tool for this approach is a local dual basis for span T_ϕ. The existence of this basis is ensured by the following theorem due to Ben-Artzi and Ron [1]. Zhao [14] followed this work and gave an alternate proof of (10). According to [1, 14], an indexed family Φ of distributions is called a *locally finite collection* if for any test function $f\in\mathscr{D}(\mathbb{R})$, $\langle\varphi, f\rangle=0$ for all but finitely many $\varphi\in\Phi$.

THEOREM 10. *Let $\Phi=\{\phi_i\}_{i\in\mathbb{Z}}$ be a locally finite collection of GLI compactly supported distributions in $\mathscr{D}'(\mathbb{R})$. Then each functional λ_i in the algebraic dual basis $\Lambda=\{\lambda_i\}_{i\in\mathbb{Z}}$ of Φ is local. Precisely, for every $i\in\mathbb{Z}$ there exists a compact $B_i\subset\mathbb{R}$ such that $\lambda_i(f)=0$ whenever* supp $f\cap B_i=\varnothing$ *and $f\in$* span *Φ.*

These sets B_i suggest the following definition of the support of the functionals λ_i.

DEFINITION 11. Let Φ and Λ be given as in Theorem 10 with $\lambda\in\Lambda$, and suppose W is an open subset of $\mathbb{R}$.

1. We say λ *vanishes on W with respect to Φ* if $\langle\lambda, \phi\rangle=0$ for all $\phi\in\operatorname{span}\Phi$ with $m(\operatorname{supp}\phi\cap W)>0$.

2. Let U be the union of all open sets $W\subset\mathbb{R}$ on which λ vanishes with respect to Φ. We define the *local support* of λ by

$$\operatorname{Lsupp}\lambda=U^c.$$

For $\lambda \in \Lambda$ and $i, k \in \mathbb{Z}$ we define $\lambda(\cdot - i)$, the ith *shift* of λ, by

$$\langle \lambda(\cdot - i), \phi_k \rangle = \langle \lambda, \phi_{k+i} \rangle,$$

and $\lambda(N^j \cdot)$, the jth *dilation* of λ, by

$$\langle \lambda(N^j \cdot), \phi_k \rangle = \langle \lambda, \phi_k(N^{-j} \cdot) \rangle.$$

The following corollary will be quite useful.

COROLLARY 12. *Suppose $\phi \in \mathscr{D}'(\mathbb{R})$ is compactly supported. Let Φ and Λ be given as in Theorem* 10 *with $\phi_i(\cdot) = \phi(\cdot - i)$ $\forall i \in \mathbb{Z}$. Then each $\lambda_i \in \Lambda$ is of the form $\lambda_i = \lambda_0(\cdot - i)$. Moreover, there exists an interval $[\alpha, \beta]$ for which* Lsupp $\lambda_0 \subset [\alpha, \beta]$.

We now set $\lambda_0 = \phi^*$ and write $\phi_{kj}(x) = \phi(N^k x - j)$ and $\phi^*_{kj}(x) = \phi^*(N^k x - j)$. It is clear that $\{\phi^*_{kj}\}$ is the dual of $\{\phi_{kj}\}$. Note that $\text{Lsupp}(\phi^*_{kj}) \subset [N^{-k}(\alpha + j), N^{-k}(\beta + j)]$. Define

$$V_k = \text{span}\{\phi_{kj}\}_{j \in \mathbb{Z}}.$$

If compactly supported ϕ satisfies (3), we have

$$\cdots \subset V_0 \subset V_1 \subset \cdots,$$

and it is known (see, e.g., [6]) that

$$\text{supp}(\phi) \subset \left[0, \frac{M}{N-1}\right].$$

LEMMA 13. *Suppose that ϕ is compactly supported and refinable, and that T_ϕ is GLI. If $f \in V_0$ and $f(x) = 0$ on $[a, b]$, then there exists an integer $k \geqslant 0$ for which*

$$f\chi_{(-\infty, a]} \in V_k \qquad \textit{and} \qquad f\chi_{[b, \infty)} \in V_k.$$

Proof. Since ϕ is compactly supported and T_ϕ is GLI, we may assume there is some $M \in \mathbb{N}$ for which ϕ satisfies (3). We may also assume WLOG that there is an $R \in \mathbb{N}$ for which $R \geqslant M/(N-1)$ and $\text{Lsupp}(\phi^*) \subset [0, R]$. Choose k so that $N^{-k}(3R+1) < b - a$. Let L denote the integer for which $N^{-k}(L-1) < a \leqslant N^{-k}L$, so

$$a \leqslant N^{-k}L < N^{-k}(L + 3R) < b. \tag{13}$$

Then from $\operatorname{Lsupp}(\phi^*_{kj}) \subset [N^{-k}j, N^{-k}(j+R)]$ we have

$$\operatorname{Lsupp}(\phi^*_{kj}) \subset [a, b], \qquad j = L, ..., L+2R. \tag{14}$$

Since ϕ is refinable, $V_0 \subset V_k$ so we can expand f as

$$f = \sum_{j \in \mathbb{Z}} c_{kj} \phi_{kj}.$$

Since $f(x) = 0$ on $[a, b]$, from (14) we have

$$c_{kl} = \sum_{j \in \mathbb{Z}} c_{kj} \delta_{jl} = \phi^*_{kl}(f) = 0$$

for $l = L+R, ..., L+2R$. Thus $f = f_1 + f_2$ where $f_1 = \sum_{j < L+R} c_{kj} \phi_{kj}$ and $f_2 = \sum_{j > L+2R} c_{kj} \phi_{kj}$ are both in V_k. Using $\operatorname{supp}(\phi) \subset [0, M/(N-1)]$, observe that $\operatorname{supp}(f_1) \subset (-\infty, N^{-k}(M/(N-1) + L)] \subset (-\infty, N^{-k}(R+L)]$ and $\operatorname{supp}(f_2) \subset [N^{-k}(L+2R), \infty)$. Thus $\operatorname{supp}(f_1) \cap \operatorname{supp}(f_2) = \emptyset$. From (13) we see that

$$f\chi_{(-\infty, a]} = f_1 \qquad \text{and} \qquad f\chi_{(b, +\infty]} = f_2,$$

which finishes the proof. ∎

Define the *symbol* of ϕ satisfying (3) in the usual way, by

$$P(z) = \frac{1}{N} \sum_{j=0}^{M} p_j z^j.$$

We may write $M = NL + w$ for some $w \in \{0, ..., N-1\}$. Define the polynomials

$$P_j(z) = p_{NL+j} + p_{N(L-1)+j} z + \cdots + p_j z^L, \tag{15}$$

where $p_i = 0$ for $i > M$.

It is known that if dilation factor $N = 2$ and T_ϕ is GLI, then the symbol of ϕ has no symmetric zeros (see, e.g., [9]). We next show this fact holds for $N \geqslant 2$ and use it to derive a property of the polynomials P_j in (15).

LEMMA 14. *If T_ϕ is GLI then*

1. *the symbol $P(z)$ of ϕ has no N-symmetric zeros, and therefore*

2. *the polynomials P_j in* (15), $w - N + 1 \leqslant j \leqslant w$, *have no common zero* $z_0 \neq 0$.

Proof. For part 1, we employ the method of [2], and suppose for the sake of contradiction thet ϕ has N-symmetric zeros

$$P(e^{-i((\omega+2k\pi)/N)})=0, \qquad k=0, ..., N-1$$

for some ω. In the transform domain, (3) becomes

$$\hat{\phi}(\omega)=\frac{1}{N}P(e^{-i\omega/N})\,\hat{\phi}\left(\frac{\omega}{N}\right).$$

Hence

$$\begin{aligned}\sum_{m\in\mathbb{Z}}|\hat{\phi}(\omega+2m\pi)|^2&=\frac{1}{N}\sum_{m\in\mathbb{Z}}|P(e^{-i(\omega+2m\pi)/N})|^2\left|\hat{\phi}\left(\frac{\omega+2m\pi}{N}\right)\right|^2\\&=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{m\in\mathbb{Z}}|P(e^{-i(\omega+2k\pi)/N})|^2\left|\hat{\phi}\left(\frac{\omega+2k\pi}{N}+2m\pi\right)\right|^2\\&=0,\end{aligned}$$

which contradicts the GLI of T_ϕ.

For part 2, suppose for the sake of contradiction that the polynomials P_j in (15) have a common zero $z_0\neq 0$. Let $z=z_0^{-1/N}$ denote any of the N roots of z_0^{-1}. Then

$$N\cdot P(z)=\sum_{j=0}^{M}p_j z^j=\sum_j z^{NL+j}P_j(z^{-N})=\sum_j z^j z_0^{-L}P_j(z_0)=0.$$

Hence $P(z)$ has N symmetric zeros, which contradicts part 1. ∎

For our next lemma, we must introduce some new notation. Recall that $M=NL+w$ for some L and $w\in\{0, ..., N-1\}$. Put

$$x_L^{(0)}=-2L-2+\frac{M}{N-1}$$

and $x_L^{(k)}=N^{-k}x_L^{(0)}$.

LEMMA 15. *Suppose that* $f\in V_{k-1}$, $f=0$ *on* $[x_L^{(k)}, N^{-k}]$, *and*

$$f\chi_{(-\infty, x_L^{(k)}]}, f\chi_{(N^{-k}, \infty]}\in V_k. \tag{16}$$

Then $f=0$ *on* $[x_L^{(k-1)}, N^{-k+1}]$ *and*

$$f\chi_{(-\infty, x_L^{(k)}]}, f\chi_{(N^{-k}, \infty]}\in V_{k-1}.$$

Proof. Without loss of generality $k=1$. From (16) we have

$$f=\sum_{j\leqslant -n} c_{1j}\phi_{1j}+\sum_{j\geqslant 1} c_{1j}\phi_{1j}, \tag{17}$$

where $n=2L+2$. Now $f\in V_0$ implies that

$$f=\sum_{j\in\mathbb{Z}} c_{0j}\phi_{0j}=\sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}} p_{j-Nk}c_{0k}\right)\phi_{1j} \tag{18}$$

and then

$$\sum_{k\in\mathbb{Z}} p_{j-Nk}\,c_{0k}=0 \tag{19}$$

for $j=-n+1, ..., 0$.

Let $(\lambda_1^j, n_1^j), ..., (\lambda_s^j, n_s^j)$ be the zeros (with multiplicities) of the polynomials $P_j(z)$ defined in (15) where $\sum_{i=1}^{s_j} n_i^j\leqslant L$ for $w-(N-1)\leqslant j\leqslant w$. From (19) and difference equation theory, there are scalars a_{it}^j for which

$$c_{0k}=\sum_{i=1}^{s_j}\sum_{t=0}^{n_i^j-1} a_{it}^j k^t(\lambda_i^j)^k \tag{20}$$

for $k=-n+1, ..., -1$, and $j=w-N+1, ..., w$. Before continuing, we need the following.

Claim. $c_{0k}=0$ *for* $k=-n+1, ..., -1$.

Proof of Claim. From $\sum_{i=1}^{s_j} n_i^j\leqslant L$ we have $n-1=2L+1\geqslant 2\sum_{i=1}^{s_j} n_i^j$. Also, for any u, v where $w-N+1\leqslant u, v\leqslant w$ we have

$$\sum_{i=1}^{s_u}\sum_{t=0}^{n_i^u-1} a_{it}^u k^t(\lambda_i^u)^k=\sum_{i=1}^{s_v}\sum_{t=0}^{n_i^v-1} a_{it}^v k^t(\lambda_i^v)^k$$

for $k=-n+1, ..., -1$ from (20). Suppose for the sake of contradiction that not all the c_{0k} are zero. We may assume that $\lambda_i^u\neq 0$ for all u, i, and, for each pair (u, i), we may assume that $a_{it}^u\neq 0$ for some t. Then by Proposition 9,

$$\sum_{i=1}^{s_u}\sum_{t=0}^{n_i^u-1} a_{it}^u x^t(\lambda_i^u)^x=\sum_{i=1}^{s_v}\sum_{t=0}^{n_i^v-1} a_{it}^v x^t(\lambda_i^v)^x$$

for all x. Hence $\lambda_i^u=\lambda_i^v$ and the polynomials (15), $w-(N-1)\leqslant j\leqslant w$, have a common zero. This contradicts Lemma 14. Thus we have $c_{0k}=0$ for $k=-n+1, ..., -1$. This completes the proof of the claim.

Now from (18) we have

$$f = \sum_{j \leqslant -n} c_{0j} \phi_{0j} + \sum_{j \geqslant 1} c_{0j} \phi_{0j},$$

where

$$\operatorname{supp}\left(\sum_{j \leqslant -n} c_{0j} \phi_{0j} \right) \subset (-\infty, x_L^{(0)}]$$

and

$$\operatorname{supp}\left(\sum_{j \geqslant 1} c_{0j} \phi_{0j} \right) \subset [N^0, \infty).$$

Hence

$$f\chi_{(-\infty, x_L^{(1)}]} = f\chi_{(-\infty, x_L^{(0)}]} \in V_0$$

and

$$f\chi_{[N^{-1}, \infty)} = f\chi_{[N^0, \infty)} \in V_0. \quad \blacksquare$$

LEMMA 16. *If* $f \in V_0$ *and* $f = 0$ *on some interval* $[a, b]$, *then*

$$f\chi_{[b, \infty)},\ f\chi_{(-\infty, a]} \in V_0.$$

Proof. Without loss of generality we assume that $a \leqslant 0$ and $b > 0$. Choose k so that $a \leqslant x_L^{(k)}$ and $N^{-k} \leqslant b$. Now using Lemmas 13 and 15 and an inductive argument, we obtain the desired result. $\blacksquare$

We are finally ready for a proof of Theorem 4.

Proof of Theorem 4. Since compactly supported ϕ satisfies (3) and T_ϕ is GLI, Theorem 2 says that ϕ has minimal convex support in span T_ϕ.

1. As noted in the Introduction, Chui and Wang [3] have shown that the convex closure of supp(ϕ) is $[0, M/(N-1)]$. We need only prove that there is no nontrivial interval $[a, b] \subset (0, M/(N-1))$ for which $\phi = 0$ on $[a, b]$. Suppose for the sake of contradiction that such an interval $[a, b]$ exists. Then by Lemma 16, $\phi\chi_{(-\infty, a]} \in V_0$ and $\phi\chi_{(-\infty, a]}$ has shorter support than ϕ. This contradicts the minimal convex support of ϕ.

2. Let L denote the integer for which $L < M/(N-1) \leqslant L+1$. Suppose for the sake of contradiction that T_ϕ is not locally linearly independent on all nontrivial intervals. Then there exist scalars d_k, not all zero, and nontrivial interval (a, b) for which $g(x) = \sum_{k \in \mathbb{Z}} d_k \phi(x-k) = 0$ on (a, b).

Shortening (a, b) and using an integer translate of g if necessary, we may assume $(a, b) \subset (L, L+1)$. Since $\text{supp}(\phi) = [0, M/(N-1)]$ by part 1,

$$\tilde{g}(x) = \sum_{k=0}^{L} d_k \phi(x-k)$$

vanishes on (a, b). Note that $\text{supp}(\tilde{g}) \subset [0, L + M/(N-1)]$. But then by Lemma 16,

$$\tilde{g}\chi_{(-\infty, a]}, \ \tilde{g}\chi_{[b, \infty)} \in V_0.$$

Hence one of these functions has support length less than $\frac{1}{2}(L + M/(N-1)) < M/(N-1)$. This contradicts the minimal convex support of ϕ. ▮

The following corollary concerning minimal support is immediate.

COROLLARY 17. *If ϕ is a refinable, compactly supported distribution and T_ϕ is GLI, then ϕ is minimally supported in* span T_ϕ *and* $\text{supp}(\phi)$ *is an interval.*

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